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## On the Foundation of Quantal Proposition Systems

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(Z. Naturforsch. **27 a**, 893–900 [1972]; received 18 January 1972)

We investigate the arguments recently given by Jauch and Piron that quantal proposition systems form a complete atomic lattice. Our analysis shows that the reasoning of these authors can not establish the intended result.

### 1. Introduction

In recent years great efforts have been undertaken to axiomatize quantum mechanics (QM) in the frame of the *proposition calculus*<sup>1,2</sup>. This research aims to deduce QM from a series of simpler and physically well established principles and to clear up to what extent the Hilbert space formalism of QM is uniquely determined by experience — or what other mathematical structures could be used to formulate QM. The justification of the Hilbert space formalism of QM on the basis of the proposition calculus is composed of two parts. The first part consists in elucidating the abstract lattice structure of quantum-mechanical (qm) proposition systems and in tracing back its characteristics to physical principles well confirmed by experience<sup>1,4</sup>; the second part deals with the mathematical representation of qm propositions by closed linear subspaces of an appropriate vector space and aims to justify the distinction of the Hilbert space as the only admissible representation space<sup>4,5,6</sup>. In both parts one still faces open questions and essential characteristics of the Hilbert space formalism of QM are still lacking a convincing experimental confirmation.

Now, in a recent paper<sup>7</sup>, JAUCH and PIRON have tried to substantiate the vital lattice-theoretic properties of quantal proposition systems by means of new proof-ideas and relying on the (partly) new hypothesis that every qm object is always in a homogeneous state whereas nonhomogeneous states represent only ensembles of qm objects as a whole

(in the sense of *macro-states*)<sup>8</sup>. In detail, Jauch and Piron made the following considerations.

(a) By an explicit instruction for combining special experiments, they try to prove that the propositions of a qm system form a complete lattice.

(b) They define a state concept for QM without recourse to probabilistic concepts.

(c) From the considerations (a) and (b) and the new state hypothesis they derive the atomicity of the proposition lattice.

(d) Finally, Jauch and Piron try to derive the covering law for quantal proposition systems from additional assumptions about the existence of ideal measurements of the first kind.

The present paper examines, by analysing the points (a) to (c), to what extent the reasonings of Ref. <sup>7</sup> really substantiate the asserted lattice structure of quantal proposition systems<sup>9</sup>. For this purpose we accept, *disputandi causa*, the new state hypothesis though this author does not believe in an experimental decision in favour of this hypothesis as far as it contradicts QM. Our analysis leads to the result that the argumentation of Ref. <sup>7</sup> fails to show that the propositions (0-1-observables) of a qm system form a complete, atomic lattice. Moreover it turns out that a “probability-free” introduction of homogeneous quantum states must be founded on different basic concepts than those used in Ref. <sup>7</sup>.

### 2. The Assumptions of Jauch and Piron

To keep the present paper self-contained, we will first repeat the concepts, considerations and as-

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sumptions of Ref. 7 which are necessary for our investigation.

In their axiomatic construction of the qm proposition calculus, Jauch and Piron start from two basic concepts: (1) the *Yes-No-Experiment* (YNE) which can be performed on qm systems and which allows only one of the two outcomes *yes* or *no*, and (2) the predicate "the YNE  $\alpha$  is true" which means that (in a given situation) the YNE  $\alpha$  yields the result *yes* with certainty and whose extension is assumed to be well determined in all situations. We denote YNEs by small greek letters and the set of all YNEs by  $\Omega$ . In accordance with 7 we make the following assumption.

(As 1) To every YNE  $\alpha \in \Omega$ ,  $\Omega$  contains also the *inverse* YNE  $\alpha^r$  resulting from  $\alpha$  by permuting the indicators *yes* and *no*. In addition,  $\Omega$  contains the "trivial" YNE  $I$  and the „absurd" YNE  $\emptyset$  which always yield the result *yes* or *no* respectively and which are related by  $I^r = \emptyset$  and  $\emptyset^r = I$ .

By an operative combination of arbitrarily given YNEs  $\alpha_1, \dots, \alpha_n$  of  $\Omega$ , Jauch and Piron introduce new YNEs: The YNE  $\prod_{i=1}^n \alpha_i$  consists in choosing first one of the  $n$  YNEs  $\alpha_i$  at random and in performing the selected YNE directly thereafter. In an idealizing generalisation of this construction, Jauch and Piron now make an assumption which is vital for their reasoning.

(As 2) Let  $\{\alpha_m; m \in M\}$  be an arbitrary set of YNEs of  $\Omega$ ; then the YNE  $\prod_{m \in M} \alpha_m$  is also contained in  $\Omega$ .

From the construction of these "product-YNEs" it follows that

$$(\prod_{m \in M} \alpha_m)^r = \prod_{m \in M} \alpha_m^r.$$

By means of the predicate ' $\alpha$  is true', Jauch and Piron then introduce a partial preorder relation  $\leq$  in  $\Omega$ , defining

$$\alpha \leq \beta := (\alpha \text{ is true} \Rightarrow \beta \text{ is true}). \quad (2.1)$$

This partial preorder relation entails the equivalence relation

$$\alpha \sim \beta := (\alpha \leq \beta \text{ and } \beta \leq \alpha) \quad (2.2)$$

and induces a partition of  $\Omega$  in equivalence classes  $a := \langle \alpha \rangle := \{\gamma \in \Omega; \gamma \sim \alpha\}$ . We call these equivalence classes *Jauch-Piron-propositions* (JPPs)<sup>10</sup> and denote the JPPs by small latin letters and the

set of all JPPs by  $\mathcal{L}$ . Finally, the partial preorder relation  $\leq$  induces a partial order relation

$$a \subset b := (\exists \alpha \in a) (\exists \beta \in b) \alpha \leq \beta \quad (2.3)$$

in  $\mathcal{L}$ . It follows from (As 2) that, for an arbitrary set  $\{\alpha_m; m \in M\}$  of YNEs of  $\Omega$ , the JPP  $\langle \prod_{m \in M} \alpha_m \rangle$  is element of  $\mathcal{L}$ , is independent of the selected representatives  $\alpha_m$  of the equivalence classes  $a_m = \langle \alpha_m \rangle$  and is the  $\subset$ -infimum of the set  $\{a_m; m \in M\}$ ; hence we get

$$\langle \prod_{m \in M} \alpha_m \rangle = \bigcap_{m \in M} \langle \alpha_m \rangle. \quad (2.4)$$

One can also show that the element  $\bigcap_{(\forall m) a_m \subset x} x$  exists

in  $\mathcal{L}$  and is the  $\subset$ -supremum of the set  $\{a_m; m \in M\}$ . So we arrive at

*Theorem 1.*  $(\mathcal{L}, \cap, \cup)$  is a complete lattice.

Obviously, the predicate ' $\alpha$  is true' is a  $\sim$ -class property and hence can be stated directly of JPPs. In accordance with their new state hypothesis, Jauch and Piron now define the *state* of a qm system as the set  $S := \{x \in \mathcal{L}; x \text{ is true}\}$  of all JPPs which are true (in the situation referred to), and about these states they make the following assumptions.

(As 3) All states are equally "detailed", that is:  $S_1 \subseteq S_2 \Rightarrow S_1 = S_2$ .

(As 4) For any  $x \in \mathcal{L}$ ,  $x \neq \langle \emptyset \rangle$ , there exists at least one state  $S$  with  $x \in S$ .

From the above definitions and assumptions one immediately gets further properties of these states:

$$x \in S, x \subset y \Rightarrow y \in S, \quad (2.5)$$

$$x \in S, y \in S \Rightarrow x \cap y \in S, \quad (2.6)$$

$$(\forall m \in M) x_m \in S \Rightarrow \bigcap_{m \in M} x_m \in S, \quad (2.6')$$

$$(\forall S) \langle \emptyset \rangle \notin S, \langle I \rangle \in S. \quad (2.7)$$

(As 3), (As 4) and Eq. (2.6') yield

*Theorem 2.*  $(\mathcal{L}, \cup, \cap)$  is atomic and, for all states  $S$ , the JPP  $\bigcap_{x \in S} x$  is an atom contained in  $S$ .

*Definition.* We denote two elements  $a, b$ , as *compatible complements* of each other if they satisfy the conditions

$$a \cup b = \langle I \rangle, a \cap b = \langle \emptyset \rangle, (\exists \alpha \in a) \alpha^r \in b.$$

In accordance with 7, we finally make the following assumptions.

(As 5) To every JPP  $x \in \mathcal{L}$  there exists exactly one compatible complement  $x'$  in  $\mathcal{L}$ .

(As 6) If  $a \subset b$ , then the sublattice generated by  $a, a', b, b'$  is Boolean.

Corollary<sup>11</sup> (As 5) and (As 6) imply the orthomodularity of the lattice  $(\mathcal{L}, \cup, \cap, ')$ .

### 3. Objections

Provided that all assumptions of Sect. 2 could be verified and all concepts based thereupon would be adequate to QM, then the above results would virtually establish the lattice structure of the proposition system of QM. But the following two objections show that the considerations of the last section can not accomplish such a substantiation.

*First Objection.* The relation  $\leq$  introduced in Sect. 2 is not equivalent to the partial preorder relation between YNEs existing in QM. Hence the JPPs defined as  $\sim$ -equivalence classes of YNEs can not be interpreted as observables of QM.

*Argument 1A.* For our first argument we introduce the concept of the *expectation value*  $E(\alpha, \pi)$  of a YNE  $\alpha$  performed on an object with the preparation  $\pi$ . In QM, all YNEs have well determined expectation values depending on the preparation of the object, and the partial preorder relation  $\sqsubseteq$  and the equivalence relation  $\sqsim$  between YNEs, valid in QM, are given by the equations

$$\alpha \sqsubseteq \beta \Leftrightarrow (\forall \pi) E(\alpha, \pi) \leq E(\beta, \pi), \quad (3.1)$$

$$\alpha \sqsim \beta \Leftrightarrow (\forall \pi) E(\alpha, \pi) = E(\beta, \pi). \quad (3.2)$$

But since the predicate ' $\alpha$  is true' (relative to the preparation  $\pi$ ) is equivalent to  $E(\alpha, \pi) = 1$ , both the two partial preorder relations  $\leq, \sqsubseteq$  and the corresponding equivalence relations  $\sim, \sqsim$  differ clearly as can be seen in the following examples<sup>12</sup>.

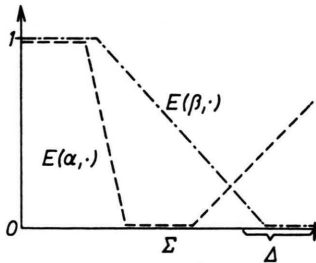


Fig. 1. Two expectation value functions  $E(\alpha, \cdot)$  and  $E(\beta, \cdot)$  are plotted over their domain  $\Sigma$ .

*Example (1).* Let us denote the set of all homogeneous preparations of a given object by  $\Sigma$ ; then the assumptions of Jauch and Piron cover also the case described in Fig. 1 where we have on one hand

$$(\forall \pi \in \Sigma) (E(\alpha, \pi) = 1 \Rightarrow E(\beta, \pi) = 1), \quad (3.3)$$

or equivalently  $\alpha \leq \beta$ , but

$$(\forall \pi \in \Delta) E(\alpha, \pi) > E(\beta, \pi) \quad (3.4)$$

on the other.

*Example (2).* In analogy to the combination  $\alpha \cdot \beta$  introduced in Sect. 2, we construct a whole family of similar measuring devices. The YNE  $\alpha[u, v] \beta$  with  $u, v \in (0, 1)$ ,  $u + v = 1$  consists of two steps: First a "stochastic decision mechanism" with two outcomes of the probabilities  $u$  and  $v$  decides between the YNEs  $\alpha$  and  $\beta$ ; subsequently the selected YNE is performed. This construction implies the relations

$$(\forall u, v) \alpha[u, v] \beta \sim \alpha[\frac{1}{2}, \frac{1}{2}] \beta = \alpha \cdot \beta \quad (3.5)$$

and

$$(\forall u, v, \pi) E(\alpha[u, v] \beta, \pi) = u E(\alpha, \pi) + v E(\beta, \pi); \quad (3.6)$$

but according to Eq. (3.6), the equation

$$(\forall u, v) \alpha[u, v] \beta \sqsubseteq \alpha \cdot \beta,$$

analogous to Eq. (3.5), is wrong.

*Example (3).* We consider two arbitrary YNEs  $\alpha, \beta$  and a preparation  $\pi$  subject to the only condition  $E(\alpha, \pi) \neq E(\beta, \pi)$ . If we assume, without loss of generality,  $E(\alpha, \pi) \leq E(\beta, \pi)$ , then we get

$$E(\alpha \cdot \beta, \pi) = \frac{1}{2} E(\alpha, \pi) + \frac{1}{2} E(\beta, \pi) \geq E(\alpha, \pi); \quad (3.7)$$

but on the other hand we have per construction  $\alpha \cdot \beta \leq \alpha$ .

*Example (4).* From the construction of the YNEs  $\alpha \cdot \beta$  and  $\alpha^\nu$ , it follows that

$$(\forall \alpha, \pi) E(\alpha \cdot \alpha^\nu, \pi) = \frac{1}{2}. \quad (3.8)$$

This yields immediately

$$\alpha \cdot \alpha^\nu = \alpha^\nu \cdot \alpha \sim \emptyset \quad (3.9)$$

whereas the analogical equation  $\alpha \cdot \alpha^\nu \sqsubseteq \emptyset$  is obviously wrong.

The Examples (3) and (4) are, in contrast to the Examples (1) and (2), not only covered but explicitly required by the assumptions of Sect. 2 and hence show especially clear the discrepancy between the pairs of relations  $\leq, \sim$  and  $\sqsubseteq, \sqsim$ .

*Argument 1 B.* The involution  $\alpha \rightarrow \alpha''$  is an orthocomplementation of the partial preorder  $(\Omega, \sqsubset)$  but not of the partial preorder  $(\Omega, \leq)$  as the equation

$$\alpha \leq \beta \Leftrightarrow \alpha'' \geq \beta'' \quad (3.10)$$

is in general not correct (see Example (1)). In particular the relation

$$\alpha \sim \beta \Leftrightarrow \alpha'' \sim \beta'', \quad (3.11)$$

self-evident for all YNEs corresponding to 0-1-observables, is not valid in  $(\Omega, \leq, '')$  as is shown by the following example.

*Example (5).* For all  $\alpha \in \Omega$ , we have  $\alpha \cdot \alpha'' \sim \emptyset$ . But the corresponding relation  $(\alpha \cdot \alpha'')'' \sim \emptyset''$  is wrong because of

$$(\alpha \cdot \alpha'')'' = \alpha'' \cdot (\alpha'')'' = \alpha'' \cdot \alpha = \alpha \cdot \alpha'' \quad (3.12)$$

and  $I = \emptyset'' \not\sim \emptyset$ .

*Second Objection.* The expectation value function of a YNE  $\prod_{m \in M} \alpha_m$  differs (for  $|M| > 1$ ) so clearly from the expectation value functions of all 0-1-observables of QM that the YNEs  $\prod_m \alpha_m$  can not be interpreted as measurements of qm observables. Hence the existence of the YNEs  $\prod_m \alpha_m$  proves nothing about the existence of dichotomic observables.

*Argument.* Per construction, the expectation value function of the YNE  $\alpha \cdot \beta$  satisfies the equation

$$(\forall \pi) E(\alpha \cdot \beta, \pi) = \frac{1}{2} E(\alpha, \pi) + \frac{1}{2} E(\beta, \pi). \quad (3.13)$$

Now let  $\alpha, \beta$  be two YNEs associated by QM to the projection operators  $P_\alpha, P_\beta$ , and let us assume that there exists a 0-1-observable which is measured by  $\alpha \cdot \beta$  and represented by a projection operator  $P_{\alpha \cdot \beta}$ . If we consider in particular the homogeneous preparations described in QM by state vectors, then the Eq. (3.13) yields, according to QM,

$$(\forall \psi) \langle \psi | P_{\alpha \cdot \beta} | \psi \rangle = \frac{1}{2} \langle \psi | P_\alpha | \psi \rangle + \frac{1}{2} \langle \psi | P_\beta | \psi \rangle.$$

But since in case of  $P_\alpha \neq P_\beta$  no projection operator  $P_{\alpha \cdot \beta}$  satisfies this equation, the YNE  $\alpha \cdot \beta$  does not measure any qm observable<sup>13</sup>.

The above objections demonstrate that the system of JPPs developped in Ref. 7 can not serve as a part of an axiomatic construction of QM. This result, on the other side, raises the question what physically correct interpretation can be given to the ele-

gant and mathematically consistent axiom system of Sect. 2 and how the system of JPPs is related to QM. Now, a reflection upon the physical significance of the JPPs shows that the JPPs are very similar to if not identical with the "propositions" on which BIRKHOFF<sup>14</sup> based his review of "quantum logic" and which he interpreted as "predictions with certainty (= probability one)" of the form  $\psi \in L_P$  where  $\psi$  denotes a state vector and  $L_P$  the eigenspace of the projection operator  $P$ . For two YNEs  $\alpha, \beta$  are  $\sim$ -equivalent if and only if they do not differ in the property ' $\xi$  is true', i.e. with regard to a certain positive outcome; all the other properties of  $\alpha$  and  $\beta$  however, and in particular the expectation values of measurements with an uncertain outcome, are left out of account by the relations  $\leq$  and  $\sim$ .

If one realizes this meaning of the JPPs, then the considerations of Ref. 7 receive a new and convincing import: they deal with axiomatizing the *certain predictions* occurring in QM. In particular, the YNE  $\alpha \cdot \beta$  can now be interpreted without hesitation as a measuring device of the conjunction of the two certain predictions  $\langle \alpha \rangle$  and  $\langle \beta \rangle$ ; for  $\alpha$  and  $\beta$  yield both a positive result with certainty if and only if  $\alpha \cdot \beta$  does so, too<sup>15</sup>. Hence Ref. 7 actually proves that the certain predictions occurring in QM form a lattice (and even a complete atomic lattice if one accepts the new state hypothesis and the idealisation of a random choice from an arbitrary large set). But since JPPs are not qm observables, this result does not have the consequences intended by Jauch and Piron for the extension and structure of the set of all qm 0-1-observables and thus does not contribute to the substantiation of the lattice structure of QM. It is, for example, easily possible that two 0-1-observables  $\mathfrak{P}, \mathfrak{Q}$ , represented in QM by the projection operators  $P, Q$ , can be measured by means of the YNEs  $\alpha, \beta$  but that no YNE is known which permits the measurement of the observable  $\mathfrak{P} \cap \mathfrak{Q}$  corresponding to the projection operator  $P \wedge Q$ ; on the other hand, the YNE  $\alpha \cdot \beta$  exists independently of the possibility to measure  $\mathfrak{P} \cap \mathfrak{Q}$  and allows one to ascertain whether the observables  $\mathfrak{P}$  and  $\mathfrak{Q}$  are both *true*.

#### 4. Discussion

In the last section it has been shown that any axiomatic construction of QM based on the pro-



positional calculus has to avoid the two objections made above. Hence we will consider in the following whether and how far our objections can be overcome by a modification of the assumptions of Ref. 7.

Let us begin with the second objection. Does there exist a general construction plan for a (realizable) combination of two YNEs which permits us to measure the genuine *quantum mechanical conjugation* of the corresponding 0-1-observables? A general impossibility-proof of such a construction plan seems without prospects. But though we know of approximate constructions<sup>16</sup>, we can give a negative answer in part: As will be shown, there exists no construction plan (operative definition) for YNEs  $\alpha * \beta$  which (1) measure the qm conjugation of the 0-1-observables measured by  $\alpha$  and  $\beta$  and which (2) consist, similar to  $\alpha \cdot \beta$ , of two steps the first of which makes a decision between  $\alpha$  and  $\beta$  on a given method while the second consists of performing the selected YNE. For such a plan would imply the existence of a function  $\Phi$  with the property

$$(\forall \alpha, \beta, \pi) E(\alpha * \beta, \pi) = \Phi[\alpha, \beta, E(\alpha, \pi), E(\beta, \pi)]. \quad (4.1)$$

But as the following example shows, this equation can not be satisfied for all YNEs and preparations considered in QM.

*Example (6).* Let  $\mathcal{H}$  be a separable Hilbert space of at least four dimensions, let  $R_1, \dots, R_4$  be four orthogonal projection operators of  $\mathcal{H}$  and  $q_j$  normed eigenvectors of  $R_j$ . We consider further two YNEs  $\alpha$  and  $\beta$  corresponding to the projection operators  $P_\alpha := R_1 + R_2$  and  $P_\beta := R_1 + R_3$  and two homogeneous preparations  $\pi_1, \pi_2$  which correspond to the state vectors

$$\begin{aligned} \psi_1 &:= (2|c_1|^2 + 2|c_2|^2)^{-\frac{1}{2}} \\ &\quad \{c_1 q_1 + c_2 q_2 + c_2 q_3 + c_1 q_4\}, \\ \psi_2 &:= (2|c_1|^2 + 2|c_2|^2)^{-\frac{1}{2}} \\ &\quad \{c_2 q_1 + c_1 q_2 + c_1 q_3 + c_2 q_4\} \end{aligned}$$

with  $c_1, c_2 \in \mathbb{R}$ ,  $c_1 \neq c_2$ . Now, if  $\alpha * \beta$  would correspond to the qm conjugation  $P_\alpha \wedge P_\beta = R_1$ , then it must hold in particular for  $i = 1, 2$

$$\begin{aligned} \langle \psi_i | P_\alpha \wedge P_\beta | \psi_i \rangle \\ = \Phi[\alpha, \beta, \langle \psi_i | P_\alpha | \psi_i \rangle, \langle \psi_i | P_\beta | \psi_i \rangle]. \end{aligned} \quad (4.2)$$

But these equations can not be satisfied because of  $\langle \psi_1 | R_1 | \psi_1 \rangle \neq \langle \psi_2 | R_1 | \psi_2 \rangle$  on one hand and

$\langle \psi_1 | P_\gamma | \psi_1 \rangle = \langle \psi_2 | P_\gamma | \psi_2 \rangle$  for  $\gamma = \alpha, \beta$  on the other. Hence, to this authors knowledge, the second objection can only be overcome by excluding the YNEs  $\prod_i \alpha_i$  "without compensation" from the set of YNEs which correspond to 0-1-observables.

The Argument 1B for the first objection can easily be overcome by subjecting the elements of  $\Omega$  to the supplementary condition

$$\alpha \leq \beta \Leftrightarrow \beta^\nu \leq \alpha^\nu \quad (4.3)$$

well confirmed by experiment.

If the axiom system of Sect. 2 is modified, according to the above considerations, by the additional postulate (4.3) and by eliminating YNEs like  $\prod_i \alpha_i$ , then the "concrete" Examples (2) to (5) are cancelled automatically and the first objection can further rely only on "formal" examples like that of Fig. 2 which are covered but no longer implied by

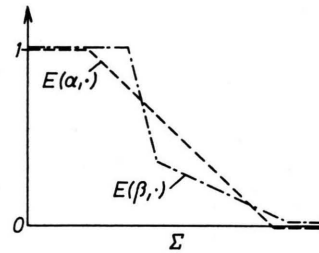


Fig. 2. In contrast to Fig. 1, here both  $\alpha \leq \beta$  and  $\beta^\nu \leq \alpha^\nu$  hold true.

the modified assumptions. Now, these examples with the property

$$(\exists \alpha, \beta, \pi) E(\alpha, \pi) > E(\beta, \pi) \text{ and } \alpha \leq \beta \quad (4.4)$$

suggest the following argument: If one restricts oneself to such YNEs and preparations which are relevant for QM by corresponding to projection operators and state operators respectively, then the two partial preorder relations  $\leq$  and  $\sqsubset$  are known to be equivalent. In consequence, all examples with the property (4.4) disagree quantitatively with QM and it should hence be possible to annul them, too, by further assumptions.

This argument is certainly true; but as the following example shows, a definite exclusion of all models of the modified assumptions which disagree with QM can not be achieved by further conditions on the structure of  $(\mathcal{L}, \subset, ')$  alone but only by taking in addition the expectation values of the YNEs into account.

*Example (7).* We consider a separable Hilbert space and denote the set of its projection operators by  $\mathcal{P}$ , the set of its elementary projection operators by  $\mathcal{E}$ , the identity (zero) operator by  $1$  ( $0$ ) and  $\mathcal{P} \setminus \{0, 1\}$  by  $\mathcal{P}_0$ . Next we introduce the set

$$\begin{aligned}\Omega_1 &:= (\mathcal{P}_0 \times \{\mathbf{a}, \mathbf{b}\}) \cup \{0, 1\} \\ &= \{Q_r; Q \in \mathcal{P}_0, r \in \{\mathbf{a}, \mathbf{b}\}\} \cup \{0, 1\}\end{aligned}$$

and the function  $E_1: \Omega_1 \times \mathcal{E} \mapsto [0, 1]$  given by

$$(\forall P \in \mathcal{E}) E_1(0, P) = 0, \quad E_1(1, P) = 1$$

$$E_1(Q_r, P) = \begin{cases} \text{Tr}(QP) \\ \text{Tr}(QP) + \frac{1}{2} \sin[2\pi \text{Tr}(QP)] \end{cases} \text{ if } r = \mathbf{a} \\ \text{if } r = \mathbf{b}.$$

In  $\Omega_1$  we define an involution  $\alpha \rightarrow \alpha^{r_1}$  by

$$1^{r_1} = 0, \quad 0^{r_1} = 1, \quad Q_r^{r_1} = (1 - Q)_r$$

and two partial preorder relations  $\leq_1$  and  $\sqsubset_1$  by

$$\alpha \leq_1 \beta \Leftrightarrow (\forall P \in \mathcal{E}) (E_1(\alpha, P) = 1 \Rightarrow E_1(\beta, P) = 1)$$

and

$$\alpha \sqsubset_1 \beta \Leftrightarrow (\forall P \in \mathcal{E}) E_1(\alpha, P) < E_1(\beta, P).$$

Obviously, the triple  $(\Omega_1, \leq_1, {}^{r_1})$  is a model of our modified axiom system. In addition, Example (7) has some interesting properties for our discussion:

(a) The orthocomplemented partial ordering  $(\mathcal{L}_1, \sqsubset_1, {}^{r_1})$  which results from  $(\Omega_1, \leq_1, {}^{r_1})$  by passing over to  $\sim_1$ -equivalence classes in analogy to Sect. 2, is isomorphic to the lattice of all closed linear subspaces of a separable Hilbert space and thus complies already with all requirements which can reasonably be made on the lattice structure of the propositions in the frame of an axiomatic construction of QM.

(b) On the other hand, Example (7) satisfies Eq. (4.4) and neither the relations  $\leq_1$  and  $\sqsubset_1$  nor the relations  $\sim_1$  and  $\sqsubset_1$  are equivalent. And as long as the expectation value  $E_1$  is not a  $\sim_1$ -class-function, the equivalence between  $\sqsubset_1$  and  $\leq_1$  can not be enforced by further requirements on  $(\mathcal{L}_1, {}^{r_1})$ .

Therefore, if one wants to found the axiomatic construction of QM on the partial preorder  $(\Omega, \leq)$  as the basic concept and to define the propositions as  $\sim$ -equivalence classes, then one has also to introduce the expectation value of YNEs in order to guarantee that it is a  $\sim$ -class function; otherwise one does not arrive at the correct proposition concept. A "probability-free" definition of (homo-

geneous) quantum states consequently requires one to start from different basis concepts, e.g. to take the proposition as a *primitive* concept (which means remove the abstraction from the YNEs to the propositions into the physical interpretation of the primitive term "proposition")<sup>17</sup>.

Thus we have shown how the assumptions of Ref. 7 must be changed in order to become acceptable as part of an axiomatic construction of QM. These changes, however, destroy most of the results which Jauch and Piron have deduced from their assumptions.

*Remark.* The concrete examples given in Sect. 3 in support of the two objections partly presuppose a special (though very plausible) interpretation of the verbal construction plan of the YNE  $\alpha \cdot \beta$ ; according to this interpretation, the selection of  $\alpha$  and of  $\beta$  in the first phase of the YNE  $\alpha \cdot \beta$  have the same probability which implies

$$(\forall \pi) E(\alpha \cdot \beta, \pi) = \frac{1}{2} E(\alpha, \pi) + \frac{1}{2} E(\beta, \pi). \quad (3.13)$$

Now, in a discussion on the first version of this paper, Dr. M. DRIESCHNER has pointed out that this interpretation of the YNE  $\alpha \cdot \beta$  may be too narrow, and he asked the question to what extent the two objections of Sect. 3 depend on this special interpretation and what changes if the "at random-selection" is interpreted as an "arbitrary" selection.

As will be shown right away, the interpretation of the construction plan of the YNE  $\alpha \cdot \beta$  does not affect the results of the present paper. If the introduction of YNEs and the use of the term "experiment" for the process  $\alpha \cdot \beta$  should make sense, then the "at random-selection" between  $\alpha$  and  $\beta$  must satisfy, in this authors opinion, at least two conditions:

(a) The YNE  $\alpha \cdot \beta$  should be reproducible and should lead to a verifiable result. Accordingly, the method of selection between  $\alpha$  and  $\beta$  should cause a well-determined expectation value  $E(\alpha \cdot \beta, \pi)$ , and this implies the existence of a function  $\Phi$  with the property

$$(\forall \pi, \alpha, \beta) E(\alpha \cdot \beta, \pi) = \Phi[\alpha, \beta, E(\alpha, \pi), E(\beta, \pi)]. \quad (4.5)$$

(b) For all YNEs  $\alpha \cdot \beta$  with  $\alpha \neq \beta$ , the method of selection between  $\alpha$  and  $\beta$  should take into consideration both YNEs with a non-zero probability:

$$\begin{aligned}(\forall \pi) [E(\alpha, \pi) \neq E(\beta, \pi) \Rightarrow \\ \min\{E(\alpha, \pi), E(\beta, \pi)\} \leq E(\alpha \cdot \beta, \pi) \\ \leq \max\{E(\alpha, \pi), E(\beta, \pi)\}].\end{aligned} \quad (4.6)$$

If we now assume, instead of Eq. (3.13), only the two conditions (4.5) and (4.6), then some equations in the examples of Sect. 3 must be modified. In detail, the following modifications would be necessary:

$$\text{Eq. (3.5)} \rightsquigarrow (\forall u, v) \alpha[u, v] \beta \sim \alpha \cdot \beta, \quad (3.5)^*$$

$$\text{Eq. (3.7)} \rightsquigarrow E(\alpha \cdot \beta, \pi) \geq E(\alpha, \pi), \quad (3.7)^*$$

$$\text{Eq. (3.8)} \rightsquigarrow (\forall \alpha, \pi) E(\alpha \cdot \alpha^\nu, \pi) \geq 0, \quad (3.8)^*$$

$$\text{Eq. (3.9)} \rightsquigarrow \alpha \cdot \alpha^\nu \sim \alpha^\nu \cdot \alpha \sim \emptyset \quad (3.9)^*$$

$$\text{Eq. (3.12)} \rightsquigarrow (\alpha \cdot \alpha^\nu)^\nu = \alpha^\nu \cdot \alpha \sim \alpha \cdot \alpha^\nu, \quad (3.12)^*$$

$$\text{Eq. (3.13)} \rightsquigarrow \text{Eq. (4.5)};$$

in addition, the refutation of Eq. (3.13) would have to be replaced by Example (6) which shows that Eq. (4.5) can not be satisfied in general; and finally, one would have to omit the discussion of the second objection in Section 4.

It is obvious that these modifications do not injure the arguments given by the examples in question. A fortiori, our objections do not depend on the particular interpretation of the construction of the YNE  $\alpha \cdot \beta$ . Just as little is the need for introducing the expectation value of the YNEs affected by the scope of interpretation for the YNE  $\alpha \cdot \beta$ .

## 5. Summary

Our analysis of the substantiation of the quantal proposition calculus given in <sup>7</sup> has led to the following results.

1) The JPPs defined in <sup>7</sup> can not be interpreted as qm observables because of essentially different characteristics. In consequence, the Theorems 1 and 2 of Sect. 2 imply nothing about the extension and structure of the set of all 0-1-observables of a qm system.

2) In order to be able to interpret the equivalence classes of YNEs as qm observables, essential changes in the axiom system of Jauch and Piron are necessary; in particular, the "product-YNEs"  $\prod_i \alpha_i$  must be excluded from the set of YNEs taken into account. But this exclusion automatically annuls the argument of Ref. <sup>7</sup> for the lattice structure of the set of all 0-1-observables of a qm system, not to speak of the completeness of this lattice. And since the atomicity of the JPP-lattice was deduced in <sup>7</sup> from its completeness, the argument for the atomic-

ity of the lattice of the 0-1-observables in QM is also cancelled — and that independently of the new state hypothesis erected in <sup>7</sup>.

3) Finally we have seen that an axiomatic construction of QM which aims to define the 0-1-observables of QM as  $\sim$ -equivalence classes of YNEs, can not avoid probabilistic concepts for this end. Hence a probability-free introduction of the homogeneous quantum states has to start from different basic concepts.

In conclusion we must state that important features of the qm proposition calculus still lack an immediate physical substantiation.

## Appendix

The strengthening of Axiom C, referred to in Footnote<sup>11</sup>, is necessary in order to guarantee the orthocomplementation of  $(\mathcal{L}, \cup, \cap, ')$  asserted by Jauch and Piron. For the retention of Axiom C which allows several compatible complements to every element of  $\mathcal{L}$  requires either to show that the entirety of all assumptions of <sup>7</sup> yet permits only one compatible complement to every element of  $\mathcal{L}$  or to pick out arbitrarily to every element  $x \in \mathcal{L}$  one example from the set of all compatible complements of  $x$  in order that the formulation of Axiom P becomes meaningful at all. But as we will see in the following example, the assumptions (As 1) to (As 4) and the Axioms P and C permit in fact several compatible complements to an element of  $\mathcal{L}$  and, moreover, they do not even guarantee that  $(\mathcal{L}, \cup, \cap)$  is *orthocomplementable*. Hence, in order to secure the vital and experimentally confirmed orthocomplementation of quantal proposition systems, we have strengthened Axiom C to (As 5).

*Example (8).* As the set  $\Omega_2$  of YNEs we choose the smallest set which (1) includes the six YNEs  $\emptyset, I, \alpha, \alpha^\nu, \beta$  and  $\beta^\nu$  and which (2) is closed under the formation of products and inverses introduced in (As 1) and (As 2). In  $\Omega_2$  we define a partial pre-order relation  $\leq_2$  and the corresponding equivalence relation  $\sim_2$  by the conditions

$$\alpha \sim_2 \beta \quad \text{and}$$

every two of  $\alpha, \alpha^\nu, \beta^\nu$  are  $\leq_2$ -incomparable.

The partial ordering  $(\mathcal{L}_2, \subset_2)$  resulting from these conditions has the structure of Fig. 3 with  $\mathcal{L}_2 = \{\emptyset, 1, a, b, c\}$ . The corresponding lattice  $(\mathcal{L}_2, \cup,$

$\cap$ ) contains exactly three pairs of compatible complements, namely  $(a, b)$ ,  $(a, c)$  and  $(0, 1)$ , and three states (in the sense of <sup>7</sup>), namely  $S_a = \{a, 1\}$ ,

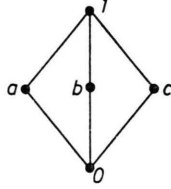


Fig. 3. The diagram of  $(\mathcal{L}_2, \subset_2)$  with  $a := \langle \alpha \rangle = \langle \beta \rangle$ ,  $0 := \langle 0 \rangle$ ,  $1 := \langle I \rangle$ ,  $b := \langle \alpha^r \rangle$  and  $c := \langle \beta^r \rangle$ .

$S_b = \{b, 1\}$  and  $S_c = \{c, 1\}$ . If we pick out a compatible complement  $a'$  of  $a$ , then the assumptions (As 1) to (As 4) and the Axioms  $C$  and  $P$  are obviously satisfied in this example whereas (As 5) is violated. And since  $\mathcal{L}_2$  has an odd number of elements, the lattice  $(\mathcal{L}_2, \cup, \cap)$  is not orthocomplementable.

The author would like to thank Prof. G. SÜSSMANN and Dr. M. DRIESCHNER for many fruitful discussions.

- <sup>1</sup> For an introduction in this complex of problems and for further references, see: J. M. JAUCH, *Foundations of Quantum Mechanics*, Addison-Wesley: Reading (Mass.) 1968.
- <sup>2</sup> By a quantum-mechanical proposition we understand an observable which can assume only one of the values 0 and 1 (i.e. an 0-1-observable) and is correspondingly represented in QM by a projection operator. Unfortunately, this usage is not generally accepted; some authors use, also in the context of QM, the term "proposition", in accordance with BIRKHOFF and von NEUMANN<sup>3</sup>, to mean a statement or a prediction. We follow the "realistic" interpretation which is also adopted by JAUCH<sup>1</sup> and PIRON<sup>4</sup>.
- <sup>3</sup> G. BIRKHOFF and J. von NEUMANN, *Ann. Math.* **37**, 823 [1936].
- <sup>4</sup> C. PIRON, *Helv. Phys. Acta* **37**, 439 [1964].
- <sup>5</sup> V. VARADARAJAN, *Geometry of Quantum Theory*, Vol. 1, van Nostrand: Princeton (N.J.) 1968.
- <sup>6</sup> S. GUDDER and C. PIRON, *J. Math. Phys.* **12**, 1583 [1971].
- <sup>7</sup> J. M. JAUCH and C. PIRON, *Helv. Phys. Acta* **42**, 842 [1969].
- <sup>8</sup> This hypothesis is new as far as it contradicts QM in that it is consequently postulated also for the correlated subsystems of a composed qm system.
- <sup>9</sup> An analysis of point (d) is given in: W. OCHS, On the Covering Law in Quantal Proposition Systems, *Commun. math. Phys.* **25**, 245 [1972].
- <sup>10</sup> In contrast to<sup>7</sup> where the  $\sim$ -equivalence classes are simply called *propositions*, we introduce here the term *JPP* in order not to anticipate a result by choosing a word. It is in fact one of the decisive arguments of this paper that the JPPs are not observables of QM.
- <sup>11</sup> Whereas (As 6) is identic with Axiom  $P$  of <sup>7</sup>, (As 5) is stronger than the corresponding assumption of <sup>7</sup> which reads: *Axiom C*. To every JPP  $x \in \mathcal{L}$  there exists at least one compatible complement  $x'$ .  
As we will demonstrate in the appendix, this strengthening of Axiom  $C$  to (As 5) is necessary for the validity of the corollary.

- <sup>12</sup> By our first objection we don't want to imply that Jauch and Piron had overlooked the discrepancy between the partial preorder relations  $\leq$  and  $\sqsubseteq$ ; our objection is simply intended to make clear that Theorem 1 (concerning the JPPs) does not prove that the 0-1-observables of a qm system form a lattice.
- <sup>13</sup> A further objection against the applicability of the YNEs  $\prod_i \alpha_i$  for the measurement of qm observables results from the following consideration: If  $\alpha$  and  $\beta$  are YNEs of the first kind, then one would expect, by reason of the unchanged character of the measurement interaction, also  $\alpha \cdot \beta$  to be an YNE of the first kind. But obviously this is not the case.
- <sup>14</sup> G. BIRKHOFF, *Lattices in Applied Mathematics*, in: *Proceedings of a Symposium in Pure Mathematics* **2**, Am. Math. Soc. **1961**, p. 157.
- <sup>15</sup> Similar considerations are found in: M. D. MACLAREN, *Notes on Axioms for Quantum Mechanics*, Report ANL-7065 of the Argonne National Laboratory [1965].
- <sup>16</sup> The possibility of approximate measuring devices (e.g. in form of alternating filter sequences) can be seen from the operator equation
 
$$\lim_{n \rightarrow \infty} Q(PQ)^n = \lim_{n \rightarrow \infty} P(QP)^n = P \wedge Q.$$
- <sup>17</sup> In this case, namely, one can first define a partial order relation  $\subset$  between the propositions by means of the predicate ' $x$  is true', and later on, after the introduction of the *expectation value of a proposition* (or of the general state), one can arrange for the two relations

$$a \text{ is true} \Rightarrow b \text{ is true}$$

$$\text{and} \quad (\forall \pi) E(a, \pi) \leq E(b, \pi)$$

to be equivalent by means of appropriate requirements on  $E$  (see e.g. Ref.<sup>1</sup>).